Homological algebra exercise sheet Week 6

1. Use the Baer Criterion to show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, and then give an injective resolution of $\mathbb{Z}$.

2. For A an abelian group, we define its *Pontrjagin dual* as :

$$A^* = \mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}/\mathbb{Z}).$$

   Show that when A is finite, we have $(A^*)^* \cong A$, and deduce that there is an isomorphism of categories $\mathrm{FinAb} \cong \mathrm{FinAb}^{\mathrm{op}}$, where FinAb is the category of finite abelian groups. However, find an abelian group such that $(A^*)^*$ is not isomorphic to A.

3. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor and $U : \mathcal{B} \to \mathcal{C}$ be an exact functor. Show that we have a natural isomorphism :

$$L_i(UF) \cong U(L_iF).$$

4. Let R be a commutative ring, and N an R-module. Show or recall that $- \otimes_R N : \text{R-}\mathbf{Mod} \to \text{R-}\mathbf{Mod}$ is right exact. We denote by $\mathrm{Tor}_i^R(-, N)$ the associated left derived functor.
   Find a projective resolution of $\mathbb{Z}/n\mathbb{Z}$, and use it to compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for every $i \geq 0$ and $m \in \mathbb{Z}$.